Mean-Variance Optimization

MGMT 675: AI-Assisted Financial Analysis

RICE | BUSINESS
Jones Graduate School of Business

Outline

- Finding the capital allocation line by solving equations
- Finding the hyperbola of efficient portolios of only risky assets
- Excel example
- Julius workflow

Solving Equations for Mean-Variance Optimization

Mean-variance optimization

- The following are essentially equivalent procedures:
    1. Maximize Sharpe ratio = risk premium / risk
    2. Maximize expected return subject to not exceeding a risk limit
    3. Minimize risk subject to achieving a target expected return
    4. Maximize expected return - penalty parameter times variance
- We can run a solver or solve equations to get the solutions.

Calculus review

- Suppose $y = x^2 - 2x + 5$ and you want to find the value of x that minimizes y.
- The minimum occurs at the bottom of the curve, where the slope is zero.
    - The slope is the derivative, and the derivative is $dy/dx = 2x - 2$.
    - So, the slope is zero when $2x - 2 = 0$, which implies $x = 1$.
- Bottom line: can solve an equation to find the minimum.

Equations that solve mean-variance problems

- w_i = weight, $\bar{r}_i$ = expected return, r_f = risk-free rate, σ_i^2 = variance, σ_{ik} = covariance.
- 3 asset example: 3 equations in 3 unknowns w_1, w_2, w_3

$$\sigma_1^2 w_1 + \sigma_{12} w_2 + \sigma_{13} w_3 = \bar{r}_1 - r_f$$

$$\sigma_{12} w_1 + \sigma_x^2 w_2 + \sigma_{13} w_3 = \bar{r}_2 - r_f$$

$$\sigma_{12} w_1 + \sigma_{12} w_2 + \sigma_3^2 w_3 = \bar{r}_3 - r_f$$

Matrix multiplication

This is not essential, but it will help to understand code.

- Three equations are represented as

$$\Sigma w = \bar{r} - r_f$$

  where $\Sigma = 3 \times 3$ array of covariances and variances, $w = (w_1, w_2, w_3)$ and $\bar{r} - r_f = (\bar{r}_1 - r_f, \bar{r}_2 - r_f, \bar{r}_3 - r_f)$

- Solution is represented as

$$w = \Sigma^{-1}(\bar{r} - r_f)$$

Tangency and other portfolios

- Given solution (w_1, w_2, w_3) of the equations, divide by sum of w_i to get tangency portfolio
- Given tangency portfolio,
  - Given risk limit (standard deviation), optimal portfolio satisfying the risk limit is

$$\frac{\text{risk limit}}{\text{std dev of tangency portfolio}} \times \text{tangency portfolio.}$$

  - Given target expected return, optimal portfolio achieving the target is

$$\frac{\text{target risk premium}}{\text{risk prem of tangency portfolio}} \times \text{tangency portfolio}$$

Excel Example

Excel Spreadsheet Example

Excel Example linked on Schedule page

based on Applied Finance Topic5.1_SharpeRatioExamples.xlsx

Risky Assets Only

Finding the Markowitz Bullet (Hyperbola)

- We can also solve equations to find
  - The global minimum variance portfolio
  - Another portfolio of risky assets on the hyperbola
- And all portfolios on the hyperbola are combinations of those two portfolios.
- However, Julius will probably run a solver if you ask it to find a portfolio of risky assets with weights summing to 1 that minimizes variance subject to a target expected return.

Julius Workflow

Julius Workflow

- User prompt:
  - Asset names, expected returns, and covariances
  - Risk-free rate
- Julius prompt:
  - Compute the tangency portfolio and output to the user
  - Compute the expected return and standard deviation of the tangency portfolio and output to the user
- Optional additional prompt for Julius:
  - Compute the minimum risk portfolio with weights that sum to 1 (fully invested in risky assets) for various expected return targets.
  - Plot the means and standard deviations of the minimum risk portfolios.
  - Show the tangency portfolio and capital allocation line on the plot.
  - Display the figure and return a jpeg of it.

Checking Julius' Work

- Run the workflow with data from Topic5.1_SharpeRatioExamples.xlsx from Applied Finance (linked on our Schedule page for Day 1).
- Check that the tangency portfolio is correct.
- If it is correct, copy the code generated by Julius and paste it into the Julius prompt cell in the workflow. Recommend that Julius use the code.